

Irreducible weight modules with a finite-dimensional weight space over the twisted $N=1$ Schrödinger-Neveu-Schwarz algebra

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Abstract. It is shown that there are no simple mixed modules over the twisted $N=1$ Schrödinger-Neveu-Schwarz algebra, which implies that every irreducible weight module over it with a nontrivial finite-dimensional weight space, is a Harish-Chandra module.

Key Words: the twisted $N=1$ Schrödinger-Neveu-Schwarz algebra, weight modules, irreducible modules

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1 Introduction

The Schrödinger-Virasoro type Lie algebras and their supersymmetric counterparts are closely related to mathematics and physics. The Schrödinger-Virasoro algebra was originally introduced by M. Henkel in 1994 during the process of trying to apply the concepts and methods of conformal field theory to models of statistical physics (see [2] for reference). The deformative counterparts were introduced in [8] and their supersymmetric extensions were investigated in the context of supersymmetric quantum mechanics. Schrödinger-Neveu-Schwarz Lie superalgebras $\mathfrak{sn}\mathfrak{s}^{(N)}$ with N supercharges were introduced in [3] generally, which appear as a semi-direct product of Lie algebras of super-contact vector fields with infinite-dimensional nilpotent Lie superalgebras. The *twisted $N=1$ Schrödinger-Neveu-Schwarz algebra* $\widetilde{\mathfrak{tsn}\mathfrak{s}}$ (introduced firstly in [3]) is an infinite-dimensional Lie superalgebra over \mathbb{C} with the basis $\{L_n, G_r, Y_p, M_p, \mathfrak{c} \mid n \in \mathbb{Z}, r \in \frac{1}{2} + \mathbb{Z}, p \in \frac{1}{2}\mathbb{Z}\}$ and the following non-vanishing super brackets:

$$[L_n, L_m] = (m - n)L_{m+n} + \frac{n^3 - n}{12}\delta_{m, -n}\mathfrak{c},$$

$$[L_n, G_r] = (r - \frac{n}{2})G_{r+n}, \quad [G_r, G_s] = 2L_{r+s} + \frac{1 - 4r^2}{12}\delta_{r, -s}\mathfrak{c},$$

$$[L_n, Y_p] = \begin{cases} (p - \frac{n}{2})Y_{p+n}, & \text{if } p \in \mathbb{Z}, \\ pY_{p+n}, & \text{if } p \in \frac{1}{2} + \mathbb{Z}, \end{cases} \quad [L_n, M_p] = \begin{cases} pM_{p+n}, & \text{if } p \in \mathbb{Z}, \\ (p + \frac{n}{2})M_{p+n}, & \text{if } p \in \frac{1}{2} + \mathbb{Z}, \end{cases}$$

$$[G_r, Y_p] = \begin{cases} \frac{1}{2}(p - r)Y_{p+r}, & \text{if } p \in \mathbb{Z}, \\ 2Y_{p+r}, & \text{if } p \in \frac{1}{2} + \mathbb{Z}, \end{cases} \quad [G_r, M_p] = \begin{cases} \frac{p}{2}M_{p+r}, & \text{if } p \in \mathbb{Z}, \\ 2M_{p+r}, & \text{if } p \in \frac{1}{2} + \mathbb{Z}, \end{cases}$$

$$[Y_p, Y_q] = \begin{cases} \frac{1}{2}(q - p)M_{q+p}, & \text{if } p, q \in \mathbb{Z}, \\ \frac{q}{2}M_{q+p}, & \text{if } p \in \mathbb{Z}, q \in \frac{1}{2} + \mathbb{Z}, \\ 2M_{q+p}, & \text{if } p, q \in \frac{1}{2} + \mathbb{Z}. \end{cases}$$

It is easy to see that $\widetilde{\mathfrak{tsns}}$ is \mathbb{Z}_2 -graded with $\widetilde{\mathfrak{tsns}} = \widetilde{\mathfrak{tsns}}_0 \oplus \widetilde{\mathfrak{tsns}}_1$, where

$$\begin{aligned}\widetilde{\mathfrak{tsns}}_0 &= \text{Span}_{\mathbb{C}}\{L_n, Y_n, M_n, \mathfrak{c} \mid n \in \mathbb{Z}\}, \\ \widetilde{\mathfrak{tsns}}_1 &= \text{Span}_{\mathbb{C}}\{G_r, Y_r, M_r \mid r \in \frac{1}{2} + \mathbb{Z}\}.\end{aligned}$$

The Cartan subalgebra (exactly the maximal toral subalgebra) of $\widetilde{\mathfrak{tsns}}$ is $\widetilde{\mathfrak{h}} = \mathbb{C}L_0 \oplus \mathbb{C}M_0 \oplus \mathbb{C}\mathfrak{c}$. It should be noted that $\widetilde{\mathfrak{tsns}}_0$ is precisely the well-known twisted Schrödinger-Virasoro Lie algebra and the subalgebra $\widetilde{\mathfrak{ns}}$ spanned by $\{L_n, G_r, \mathfrak{c} \mid n \in \mathbb{Z}, r \in \frac{1}{2} + \mathbb{Z}\}$ is the $N = 1$ Neveu-Schwarz algebra. Denote $\widetilde{\mathfrak{J}} = \mathfrak{J} \oplus \mathfrak{c}$ with $\mathfrak{J} = \text{Span}_{\mathbb{C}}\{Y_p, M_p \mid p \in \frac{1}{2}\mathbb{Z}\}$. It is easy to see that both $\widetilde{\mathfrak{J}}$ and \mathfrak{J} are ideals of $\widetilde{\mathfrak{tsns}}$.

An $\widetilde{\mathfrak{h}}$ -diagonalizable module over $\widetilde{\mathfrak{tsns}}$ is usually called *weight module*. If all weight spaces of a weight $\widetilde{\mathfrak{tsns}}$ -module are finite-dimensional, the module is called a *Harish-Chandra module*. We introduce the following notations:

$$\begin{aligned}\widetilde{\mathfrak{tsns}}_+ &= \text{Span}_{\mathbb{C}}\{L_n, Y_p, M_p, G_r \mid n \in \mathbb{Z}^+, p \in \frac{1}{2}\mathbb{Z}, r \in \frac{1}{2} + \mathbb{Z}, p, r > 0\}, \\ \widetilde{\mathfrak{tsns}}_- &= \text{Span}_{\mathbb{C}}\{L_n, Y_p, M_p, G_r \mid n \in \mathbb{Z}^-, p \in \frac{1}{2}\mathbb{Z}, r \in \frac{1}{2} + \mathbb{Z}, p, r < 0\}, \\ \widetilde{\mathfrak{tsns}}_0 &= \text{Span}_{\mathbb{C}}\{L_0, Y_0, M_0, \mathfrak{c}\}.\end{aligned}$$

Then $\widetilde{\mathfrak{tsns}}$ admits the following triangular decomposition:

$$\widetilde{\mathfrak{tsns}} = \widetilde{\mathfrak{tsns}}_+ \oplus \widetilde{\mathfrak{tsns}}_0 \oplus \widetilde{\mathfrak{tsns}}_-.$$

For any irreducible weight $\widetilde{\mathfrak{tsns}}$ -module V , L_0 , M_0 and \mathfrak{c} must act as some complex numbers on it. Furthermore, V has the weight space decomposition $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$, where $V_\lambda = \{v \in V \mid L_0 v = \lambda v\}$ is called a *weight space* with weight λ . Denote the set of *weights* λ of V by $\text{supp}(V) := \{\lambda \in \mathbb{C} \mid V_\lambda \neq 0\}$, which is called the *support* of V . If V is an irreducible weight $\widetilde{\mathfrak{tsns}}$ -module, then there exists some $\lambda \in \mathbb{C}$ such that $\text{supp}(V) \subseteq \lambda + \frac{1}{2}\mathbb{Z}$.

An irreducible weight module V is called a *pointed module* if there exists a weight $\lambda \in \mathbb{C}$ such that $\dim V_\lambda = 1$. The following natural problem was firstly referred in [10]:

Problem 1.1 *Is any irreducible pointed module over the Virasoro algebra a Harish-chandra module?*

An irreducible weight module V is called a *mixed module* if there exist $\lambda \in \mathbb{C}$ and $k \in \mathbb{Z}^*$ such that $\dim V_\lambda = \infty$ and $\dim V_{\lambda+k} < \infty$. The following conjecture was given in [6]:

Conjecture 1.2 *There are no irreducible mixed modules over the Virasoro algebra.*

The positive answers to the above question and conjecture were given in [7]. Such question and conjecture have already been solved on the truncated Virasoro algebras, the W -algebra $W(2, 2)$, the twisted Schrödinger-Virasoro algebra, the twisted Heisenberg-Virasoro

algebra and the Neveu-Schwarz algebra in [1, 4, 5, 9, 11]. In this note, we also give the positive answers to the above question and conjecture for the twisted $N = 1$ Schrödinger-Neveu-Schwarz algebra. Our main result is the following:

Theorem 1.3 *For any irreducible weight $\widetilde{\mathfrak{tsns}}$ -module V , if there exists some $\lambda \in \mathbb{C}$ such that $\dim V_\lambda = \infty$, then $\text{supp}(V) = \lambda + \frac{1}{2}\mathbb{Z}$ and $\dim V_{\lambda+k} = \infty$ for every $k \in \frac{1}{2}\mathbb{Z}$. In other words, for any irreducible weight $\widetilde{\mathfrak{tsns}}$ -module V , the condition that there exists some $\lambda \in \mathbb{C}$ such that $0 < \dim V_\lambda < \infty$ implies that V is a Harish-Chandra module. Then there are no irreducible mixed $\widetilde{\mathfrak{tsns}}$ -modules.*

Throughout this paper, we respectively denote \mathbb{Z}^* , \mathbb{Z}^+ , \mathbb{Z}^- , \mathbb{Z}_+ and \mathbb{Z}_- the sets of the nonzero, positive, negative, nonnegative and non-positive integers.

2 Proof of Theorem 1.3

We first recall a main result about the irreducible weight Neveu-Schwarz-modules given in [11], which we cited here as the following lemma:

Lemma 2.1 *Let V be an irreducible weight $\widetilde{\mathfrak{ns}}$ -module. Assume that there exists $\lambda \in \mathbb{C}$ such that $\dim V_\lambda = \infty$. Then $\text{supp}(V) = \lambda + \frac{1}{2}\mathbb{Z}$ and for every $k \in \frac{1}{2}\mathbb{Z}$, we have $\dim V_{\lambda+k} = \infty$.*

Since $\mathfrak{I} = \text{Span}_{\mathbb{C}}\{Y_p, M_p \mid p \in \frac{1}{2}\mathbb{Z}\}$ is a nontrivial ideal of $\widetilde{\mathfrak{tsns}}$, $\mathfrak{I}V$ is a submodule of V for any $\widetilde{\mathfrak{tsns}}$ -module V , which gives the following lemma (such lemma was not used in [11]).

Lemma 2.2 *For any irreducible weight $\widetilde{\mathfrak{tsns}}$ -module V , we have $\mathfrak{I}V = 0$ or $\mathfrak{I}V = V$.*

For any irreducible weight $\widetilde{\mathfrak{tsns}}$ -module V , if we can prove $\mathfrak{I}V = 0$, then V will degenerate to be an irreducible weight $\widetilde{\mathfrak{ns}}$ -module. Then Theorem 1.3 follows from Lemma 2.1 in this case. Denote the universal enveloping algebra of $\widetilde{\mathfrak{tsns}}$ by $U(\widetilde{\mathfrak{tsns}})$. For any irreducible weight $\widetilde{\mathfrak{tsns}}$ -module V , if there exists some $0 \neq v \in V$ such that $\mathfrak{I}v = 0$, then $U(\widetilde{\mathfrak{tsns}})v = V$. And the following lemma follows:

Lemma 2.3 *Assume that V is an irreducible weight module over $\widetilde{\mathfrak{tsns}}$. If there exists some $0 \neq v \in V$ such that $\mathfrak{I}v = 0$. Then \mathfrak{I} acts trivially on V and V is simply an irreducible weight module over $\widetilde{\mathfrak{ns}}$.*

It is easily to see that $\{G_{\frac{3}{2}}, G_{\frac{1}{2}}, Y_{\frac{1}{2}}, M_{\frac{1}{2}}\}$ and $\{G_{-\frac{3}{2}}, G_{-\frac{1}{2}}, Y_{-\frac{1}{2}}, M_{-\frac{1}{2}}\}$ respectively generate $\widetilde{\mathfrak{tsns}}_+$ and $\widetilde{\mathfrak{tsns}}_-$. Then the following lemma follows from the fact that both highest and lowest weight $\widetilde{\mathfrak{tsns}}$ -modules are Harish-Chandra modules (similar to the Virasoro algebra case investigated in [7]).

Lemma 2.4 *Assume that there exists $\mu \in \mathbb{C}$ and a non-zero element $v \in V_\mu$, such that*

$$G_{\frac{3}{2}}v = G_{\frac{1}{2}}v = Y_{\frac{1}{2}}v = M_{\frac{1}{2}}v = 0 \quad \text{or} \quad G_{-\frac{3}{2}}v = G_{-\frac{1}{2}}v = Y_{-\frac{1}{2}}v = M_{-\frac{1}{2}}v = 0.$$

Then V is a Harish-Chandra module.

Proof of Theorem 1.3 We shall prove this theorem step by step by several lemmas.

Assume now that V is an irreducible weight $\widetilde{\mathfrak{tsns}}$ -module such that there exists $\lambda \in \mathbb{C}$ satisfying $\dim V_\lambda = \infty$. Denote the set $S_\lambda = \{p \mid \dim V_{\lambda+p} < \infty, p \in \frac{1}{2}\mathbb{Z}^*\}$.

Lemma 2.5 *For any $\lambda \in \mathbb{C}$ satisfying $\dim V_\lambda = \infty$, there are at most two adjacent elements in S_λ . For convenience, we can suppose $S_\lambda \subseteq \{\frac{1}{2}, 1\}$.*

Proof Suppose there are two different elements p and q in S_λ . Without loss of generality, we can assume $p = \frac{1}{2}$, $q > \frac{1}{2}$.

Case 1 $p = \frac{1}{2}$, $q > \frac{1}{2}$ and $q \in \frac{1}{2} + \mathbb{Z}$.

Let W be the intersection of the kernels of the linear maps $X_{\frac{1}{2}} : V_\lambda \rightarrow V_{\lambda+\frac{1}{2}}$ and $X_q : V_\lambda \rightarrow V_{\lambda+q}$ for $X = G, Y, M$. Then $X_{\frac{1}{2}}W = X_qW = 0$ for all $X \in \{G, Y, M\}$. The assumptions $\dim V_\lambda = \infty$, $\dim V_{\lambda+\frac{1}{2}} < \infty$ and $\dim V_{\lambda+q} < \infty$ force $\dim W = \infty$. Using $G_{\frac{1}{2}}W = Y_{\frac{1}{2}}W = M_{\frac{1}{2}}W = 0$, $G_qW = Y_qW = M_qW = 0$ and the given brackets, we obtain $L_nW = G_rW = Y_pW = M_pW = 0$ for all $n = 1, 2q, 2q+1, 2q+2, \dots, r = \frac{1}{2}, q, q+1, q+2, \dots$, $p \in \frac{1}{2}\mathbb{Z}_{>0}$. We claim that $q > \frac{3}{2}$ and $G_{\frac{3}{2}}w \neq 0$ for all $0 \neq w \in W$. Otherwise, W would be a Harish-Chandra module by Lemma 2.4. Thus $\dim G_{\frac{3}{2}}W = \infty$. Since $\dim V_{\lambda+\frac{1}{2}} < \infty$, there exists some $w \in W$ such that $v = G_{\frac{3}{2}}w$ and $L_{-1}v = Y_{-1}v = M_{-1}v = 0$. It is easily to verify that $\widetilde{\mathfrak{tsns}}_+v = 0$, which forces V to be a Harish-Chandra module according to Lemma 2.4. This contracts with our assumptions. Hence this case is impossible.

Case 2 $p = \frac{1}{2}$, $q > 1$ and $q \in \mathbb{Z}$.

Let W be the intersection of the kernels of the linear maps $X_{\frac{1}{2}} : V_\lambda \rightarrow V_{\lambda+\frac{1}{2}}$ and $Z_q : V_\lambda \rightarrow V_{\lambda+q}$ for $Z = L, Y, M$. Then $X_{\frac{1}{2}}W = Z_qW = 0$ for all $X \in \{G, Y, M\}$ and $Z \in \{L, Y, M\}$. The assumptions $\dim V_\lambda = \infty$, $\dim V_{\lambda+\frac{1}{2}} < \infty$ and $\dim V_{\lambda+q} < \infty$ force $\dim W = \infty$. Using $G_{\frac{1}{2}}W = Y_{\frac{1}{2}}W = M_{\frac{1}{2}}W = 0$, $L_qW = Y_qW = M_qW = 0$ and the given brackets, we obtain $L_nW = G_rW = Y_pW = M_pW = 0$ for all $n = 1, q, q+1, q+2, \dots$, $r = \frac{1}{2}, q + \frac{1}{2}, q + \frac{3}{2}, \dots$, $p \in \frac{1}{2}\mathbb{Z}^+$. We claim that $G_{\frac{3}{2}}w \neq 0$ for all $0 \neq w \in W$. Otherwise, W would be a Harish-Chandra module by Lemma 2.4. Thus $\dim G_{\frac{3}{2}}W = \infty$. Similar to the proof of the first case, one can prove that this case is also impossible. Then this lemma follows. \square

The following lemma can be easily verified, partially given in Lemma 2 of [11].

Lemma 2.6 (i) *Let $0 \neq v \in V$ be such that $G_{\frac{1}{2}}v = Y_{\frac{1}{2}}v = M_{\frac{1}{2}}v = 0$. Then*

$$\left(\frac{1}{2}L_1G_{\frac{1}{2}} - G_{\frac{3}{2}}\right)G_{\frac{3}{2}}v = 0 = Y_pv = M_pv \text{ for all } p \in \frac{1}{2}\mathbb{Z}^+.$$

(ii) *Let $0 \neq v \in V$ be such that $G_{-\frac{1}{2}}v = Y_{-\frac{1}{2}}v = M_{-\frac{1}{2}}v = 0$. Then*

$$\left(\frac{1}{2}L_{-1}G_{-\frac{1}{2}} + G_{-\frac{3}{2}}\right)G_{-\frac{3}{2}}v = 0 = Y_{-p}v = M_{-p}v \text{ for all } p \in \frac{1}{2}\mathbb{Z}^-.$$

According to Lemma 2.5, we can assume V is an irreducible weight $\widetilde{\mathfrak{tsns}}$ -module such that there exists some $\mu \in \mathbb{C}$ satisfying $\dim V_\mu < \infty$, $\dim V_{\mu+\frac{1}{2}} < \infty$ and $\dim V_{\mu+p} = \infty$ for all $p \in \mathbb{Z}^* \cup \{\frac{1}{2} + \mathbb{Z}^*\}$.

Lemma 2.7 $\mu \in \{-1, \frac{1}{2}\}$.

Proof Let U be the intersection of the kernels of the linear maps $X_{\frac{1}{2}} : V_{\mu-\frac{1}{2}} \rightarrow V_\mu$ for $X = G, Y, M$. Then $G_{\frac{1}{2}}U = Y_{\frac{1}{2}}U = M_{\frac{1}{2}}U = 0$. The assumptions $\dim V_\mu < \infty$ and $\dim V_{\mu-\frac{1}{2}} = \infty$ force $\dim U = \infty$. We claim that $G_{\frac{3}{2}}u \neq 0$ for all $0 \neq u \in U$. Otherwise, M would be a Harish-Chandra module by Lemma 2.4. Thus $\dim G_{\frac{3}{2}}U = \infty$. Since $\dim M_\mu < \infty$, there exists some $u \in U$ such that $v = G_{\frac{3}{2}}u$ and $G_{-\frac{1}{2}}v = Y_{-\frac{1}{2}}v = M_{-\frac{1}{2}}v = 0$. Using Lemma 2.6, we obtain $(\frac{1}{2}L_1G_{\frac{1}{2}} - G_{\frac{3}{2}})v = 0$ and $Y_pu = M_pu = 0$ for all $p \in \frac{1}{2}\mathbb{Z}^+$. Then U is simply a $\widetilde{\mathfrak{ns}}$ -module. Then we can use the same discussions as those given in Lemma 3 of [11], and have the following observations:

$$0 = L_{-1}G_{-\frac{1}{2}}(\frac{1}{2}L_1G_{\frac{1}{2}} - G_{\frac{3}{2}})w = (2L_0^2 - 3L_0)w = (2(\mu+1)^2 - 3(\mu+1)).$$

Then the lemma follows. \square

According to Lemma 2.7, any irreducible weight $\widetilde{\mathfrak{tsns}}$ -module V possessing only two different finite-dimensional weight spaces can be assumed to be of the following type: $\dim V_{\frac{1}{2}} < \infty$, $\dim V_1 < \infty$ and $\dim V_{p+\frac{1}{2}} = \infty$ for all $p \in \mathbb{Z}^* \cup \{\frac{1}{2} + \mathbb{Z}^*\}$. Without loss of generality, we will just prove that this case can not happen, which can be formulated as the following lemma:

Lemma 2.8 *The irreducible weight $\widetilde{\mathfrak{tsns}}$ -module V satisfying $\dim V_{\frac{1}{2}} < \infty$, $\dim V_1 < \infty$ and $\dim V_{\frac{1}{2}+p} = \infty$ for all $p \in \mathbb{Z}^* \cup \{\frac{1}{2} + \mathbb{Z}^*\}$ does not exist.*

Proof Let W be the intersection of the kernels of the linear maps $X_{\frac{1}{2}} : V_0 \rightarrow V_{\frac{1}{2}}$ for all $X \in \{G, Y, M\}$. Then $G_{\frac{1}{2}}W = Y_{\frac{1}{2}}W = M_{\frac{1}{2}}W = L_0W = 0$ and $\dim W = \infty$. According to the following facts:

$$[G_{\frac{1}{2}}, G_{\frac{1}{2}}] = 2L_1, \quad [L_1, Y_r] = rY_{1+r}, \quad [L_1, M_r] = (r + \frac{1}{2})M_{r+1}, \quad \forall r \in \frac{1}{2} + \mathbb{Z}_+,$$

we can prove the following identities:

$$L_1W = Y_rW = M_rW = 0, \quad \forall r \in \frac{1}{2} + \mathbb{Z}_+. \quad (2.1)$$

Recalling the following identities:

$$[G_{\frac{1}{2}}, Y_p] = 2Y_{p+\frac{1}{2}}, \quad [Y_p, Y_q] = 2M_{p+q}, \quad \forall p, q \in \frac{1}{2} + \mathbb{Z},$$

we can obtain the following identities:

$$Y_n W = M_n W = 0, \quad \forall n \in \mathbb{Z}_+. \quad (2.2)$$

Then combining (2.1) and (2.2), we arrive at the following identities:

$$Y_p W = M_p W = 0, \quad \forall p \in (\frac{1}{2}\mathbb{Z}_+)^*. \quad (2.3)$$

Using the above obtained results, we can prove the following identities:

$$Y_p G_{\frac{3}{2}} W = M_p G_{\frac{3}{2}} W = 0, \quad \forall p \in (\frac{1}{2}\mathbb{Z}_+)^*. \quad (2.4)$$

We claim that $G_{\frac{3}{2}} w \neq 0$ for all $0 \neq w \in W$. Otherwise, V would become a Harish-Chandra module by Lemma 2.4, which forces $\dim G_{\frac{3}{2}} W = \infty$. Recalling our suppose $\dim V_1 < \infty$, we can deduce that there exists $0 \neq v \in G_{\frac{3}{2}} W$ such that $G_{-\frac{1}{2}} v = Y_{-\frac{1}{2}} v = M_{-\frac{1}{2}} v = 0$. Then $L_{-1} v = G_{-\frac{1}{2}} G_{-\frac{1}{2}} v = 0$. According to the following identities:

$$\begin{aligned} [L_{-1}, Y_p] &= p Y_{p-1}, \quad [L_{-1}, M_p] = (p - \frac{1}{2}) M_{p-1}, \\ [G_{-\frac{1}{2}}, Y_p] &= 2 Y_{p-\frac{1}{2}}, \quad [Y_p, Y_q] = 2 M_{p+q}, \quad \forall p, q \in \frac{1}{2} + \mathbb{Z}^-, \end{aligned}$$

we can deduce the following results:

$$Y_p v = M_p v = 0, \quad \forall p, q \in (\frac{1}{2}\mathbb{Z}_-)^*. \quad (2.5)$$

The identities given in (2.4), imply

$$Y_p v = M_p v = 0, \quad \forall p, q \in (\frac{1}{2}\mathbb{Z}_+)^*, \quad (2.6)$$

Combining $Y_0 = \frac{1}{2}[G_{-\frac{1}{2}}, Y_{\frac{1}{2}}]$, $M_0 = \frac{1}{2}[G_{-\frac{1}{2}}, M_{\frac{1}{2}}]$, (2.5) and (2.6), we can deduce

$$Y_0 v = M_0 v = 0,$$

which together with (2.5) and (2.6), gives

$$Y_p v = M_p v = 0, \quad \forall p \in \frac{1}{2}\mathbb{Z}.$$

Then recalling Lemma 2.3, we can finally deduce that Y_p, M_p act trivially on the whole V for all $p \in \frac{1}{2}\mathbb{Z}$ and V is simply an irreducible weight module over $\tilde{\mathfrak{ns}}$. Then this lemma follows from Lemma 2.1. \square

By Lemmas 2.5, 2.7 and 2.8, we get the following lemma immediately.

Lemma 2.9 *There is at most one element in S_λ .*

According to Lemma 2.9, we can suppose there exists some $p \in \frac{1}{2}\mathbb{Z}^*$, such that $\dim V_{\lambda+p} < \infty$. For convenience, we denote $\lambda + p$ by μ in the following lemma.

Lemma 2.10 $S_\lambda = \emptyset$.

Proof According to our suppose, we know $\dim V_\mu < \infty$ with $\mu \in S_\lambda$. Let W be the intersection of the kernels of the linear maps $X_{\frac{1}{2}} : V_{\mu-\frac{1}{2}} \rightarrow V_\mu$ for $X \in \{G, Y, M\}$. Then $G_{\frac{1}{2}}W = Y_{\frac{1}{2}}W = M_{\frac{1}{2}}W = L_0W = 0$ and $\dim W = \infty$. Similar to the corresponding proof of Lemma 2.8, we can prove the following identities:

$$\dim G_{\frac{3}{2}}W = \infty, \quad (2.7)$$

$$Y_p G_{\frac{3}{2}}W = M_p G_{\frac{3}{2}}W = 0, \quad \forall p \in (\frac{1}{2}\mathbb{Z}_+)^*. \quad (2.8)$$

Recalling our suppose that $\dim V_\mu < \infty$, we claim that there exists some nonzero element $v \in G_{\frac{3}{2}}W$ such that

$$L_{-1}v = Y_{-1}v = M_{-1}v = 0. \quad (2.9)$$

According to (2.8), we also know that

$$Y_p v = M_p v = 0, \quad \forall p, q \in (\frac{1}{2}\mathbb{Z}_+)^*. \quad (2.10)$$

The following identities hold:

$$[L_{-1}, Y_{1-p}] = \begin{cases} (\frac{3}{2} - p)Y_{-p}, & \text{if } p \in \mathbb{Z}, \\ (1 - p)Y_{-p}, & \text{if } p \in \frac{1}{2} + \mathbb{Z}, \end{cases}$$

from which we can deduce

$$Y_p v = 0, \quad \forall p \in \frac{1}{2}\mathbb{Z}_-. \quad (2.11)$$

Combining (2.10) and (2.12), we arrive at the following result:

$$Y_p v = 0, \quad \forall p \in \frac{1}{2}\mathbb{Z}. \quad (2.12)$$

The identity $M_p v = 0$ for all $p \in \frac{1}{2}\mathbb{Z}$ follows from 2.12 and the follows identities:

$$[Y_p, Y_q] = \begin{cases} \frac{q}{2}M_{q+p}, & \text{if } p \in \mathbb{Z}, q \in \frac{1}{2} + \mathbb{Z}, \\ 2M_{q+p}, & \text{if } p, q \in \frac{1}{2} + \mathbb{Z}. \end{cases}$$

Then again recalling Lemma 2.3, we can finally deduce that Y_p, M_p act trivially on the whole V for all $p \in \frac{1}{2}\mathbb{Z}$ and V is simply an irreducible weight module over $\tilde{\mathfrak{ns}}$. Then this lemma follows from Lemma 2.1. \square

Then Theorem 1.3 follows immediately from Lemma 2.10. \blacksquare

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References

- [1] X. Guo, X. Liu, Weight modules with a finite-dimensional weight space over the truncated Virasoro algebras, *J. Math. Phys.*, **51** (2010), 123522.
- [2] M. Henkel, Schrödinger invariance and strongly anisotropic critical systems, *J. Stat. Phys.*, **75**(1994), 1023–1029.
- [3] M. Henkel, J. Unterberger, Supersymmetric extensions of Schrödinger-invariance, *Nucl. Phys. B*, **746** (2006), 155–201.
- [4] D. Liu, S. Gao, L. Zhu, Classification of irreducible weight modules over W -algebra $W(2, 2)$, *J. Math. Phys.*, **49** (2008), 113503.
- [5] J. Li, Y. Su, Irreducible weight modules over the twisted Schrödinger-Virasoro algebra, *Acta Mathematica Sinica, English Series*, **25**(4) (2009), 531–536.
- [6] V. Mazorchuk, On simple mixed modules over the Virasoro algebra, *Mat. Stud.*, **22** (2004), 121–128.
- [7] V. Mazorchuk, K. Zhao, Classification of simple weight Virasoro modules with a finite-dimensional weight space, *J. Algebra*, **307** (2007), 209–214.
- [8] J. Unterberger, C. Roger, The Schrödinger-Virasoro Lie algebra, *Springer Texts and Monographs in Physics*, Springer (Heidelberg 2012).
- [9] R. Shen, Y. Su, Classification of irreducible weight modules with a finite-dimensional weight space over twisted Heisenberg-Virasoro algebra, *Acta Math. Sinica, English Series*, **23** (2007), 189–192.
- [10] X. Xu, Pointed representations of Virasoro algebra, A Chinese summary appears in *Acta Math. Sinica, Chinese Series*, **40**(3), 479; *Acta Math. Sinica, New Series*, **13** (1997), 161–168.
- [11] X. Zhang, Z. Xia, Classification of simple weight modules for the Neveu-Schwarz algebra with a finite-dimensional weight space, *Comm. Alg.* **40** (2012), 2161–2170.